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Recall Conditional Exp.

We are given a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ & a r.v. $\xi:\Omega\to\mathbb{R}^d$ $\mathbb{E}(|\xi|) = \int_\Omega |\xi(\omega)|\, d\mathbb{P}(\omega) < \infty$ $(\xi \in \mathcal{L}^1(\mathbb{P}))$:

[redacted]

WE HAVE SEEN:

If further:

$\xi:\Omega\to\mathbb{R}$ has a density,

$Y:\Omega\to\mathbb{R}$ is another r.v.

which has a density: $f_Y(y)$

they have a joint density:

$f_{\xi,y}(\xi,y)$

$$\mathbb{E}(\xi|Y)(\xi,y) = \frac{1}{f_Y(y)} \int_{-\infty}^{\infty} x \cdot f_{\xi,y}(\xi,y)\,dx$$

Recall Conditional Exp.

We are given a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ & a r.v.
$\xi : \Omega \to \mathbb{R}^d$ $\mathbb{E}(|\xi|) = \int_\Omega |\xi(\omega)| d\mathbb{P}(\omega) < \infty$ $(\xi \in \mathcal{L}^1(\mathbb{P}))$:

LEMMA. (wrt. to another r.v.)

The c.e. def. as on the RHS satisfy

a) $\mathbb{E}(\xi|Y) \in \sigma(Y)$

$\big(\mathbb{E}(\xi|Y) = g(Y)$ for some $g : ? \to \mathbb{R}$ Borel measurable$\big)$

b) $\mathbb{E}(\mathbb{E}(\xi|Y); H) = \mathbb{E}(\xi; H)$ for $\forall H \in \sigma(Y)$.

WE HAVE SEEN:

If further:
$\xi : \Omega \to \mathbb{R}$ has a density,
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Recall Conditional Exp.

We are given a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ & a r.v. $\xi:\Omega\to\mathbb{R}^d$ $\mathbb{E}(|\xi|)=\int_\Omega |\xi(\omega)|\,d\mathbb{P}(\omega)<\infty$ $(\xi\in\mathcal{L}^1(\mathbb{P}))$:

LEMMA. (wrt. to another r.v.)

Let Y be a r.v. $Y:\Omega\to\mathbb{R}^a$, $Y\in\mathcal{A}$, $\mathbb{E}(|Y|)<\infty$.

a) $\mathbb{E}(\xi|Y)\in\sigma(Y)$

$\big(\mathbb{E}(\xi|Y)=g(Y)$ for some $g:?\to\mathbb{R}$ Borel measurable$\big)$

b) $\mathbb{E}(\mathbb{E}(\xi|Y);A)=\mathbb{E}(\xi;A)$ for $\forall A\in\sigma(Y)$.

$\approx$

DEF. (wrt. a sub-σ-algebra)

Let $\mathcal{F}$ be a sub-σ-algebra of $\mathcal{A}$. $\mathbb{E}(\xi|\mathcal{F})$ is the random variable satisfying: $(\mathbb{E}(\xi|\mathcal{F}):\Omega\to\mathbb{R}^d)$

a) $\mathbb{E}(\xi|\mathcal{F})\in\mathcal{F}$,

b) $\forall A\in\mathcal{F}$:

$$\begin{array}{ccc}\displaystyle\int_A \xi(\omega)\,d\mathbb{P}(\omega) & = & \displaystyle\int_A \mathbb{E}(\xi|\mathcal{F})\,d\mathbb{P}(\omega) \\ \| & & \| \\ \mathbb{E}(\xi;A) & & \mathbb{E}(\mathbb{E}(\xi|\mathcal{F});A)\end{array}$$

Recall Conditional Exp.

We are given a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ & a r.v. $\xi : \Omega \to \mathbb{R}^d$ $\mathbb{E}(|\xi|) = \int_\Omega |\xi(\omega)| d\mathbb{P}(\omega) < \infty$ $(\xi \in \mathcal{L}^1(\mathbb{P}))$:

DOES THIS DEF MAKES SENSE?

A/ It exists for $\xi \in \mathcal{A}$, $\mathcal{F} \subseteq \mathcal{A}$, $\mathbb{E}(|\xi|) < \infty$

B/ It is unique in some sense

C/ It represents our intuition in simple cases.

DEF. (wrt. a sub-σ-algebra)

Let $\mathcal{F}$ be a sub-σ-algebra of $\mathcal{A}$. $\mathbb{E}(\xi|\mathcal{F})$ is the random variable satisfying:

a) $\mathbb{E}(\xi|\mathcal{F}) \in \mathcal{F}$,

b) $\forall A \in \mathcal{F}$:

$$\underbrace{\int_A \xi(\omega) d\mathbb{P}(\omega)}_{= \mathbb{E}(\xi; A)} = \underbrace{\int_A \mathbb{E}(\xi|\mathcal{F}) d\mathbb{P}(\omega)}_{= \mathbb{E}(\mathbb{E}(\xi|\mathcal{F}); A)}$$

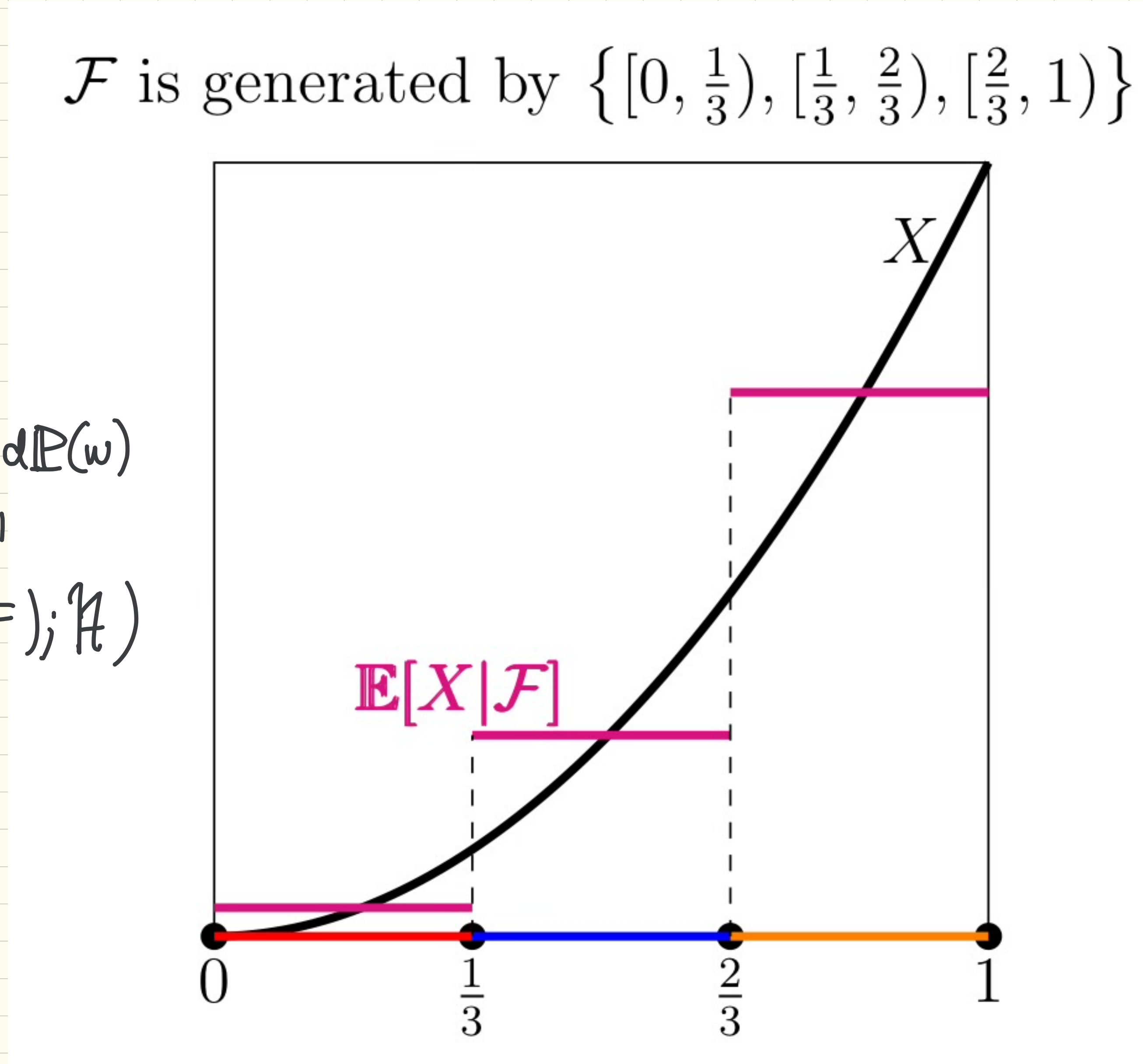
Examples.

a) $\mathbb{E}(\xi|\mathcal{F}) \in \mathcal{F}$,

b) $\forall A \in \mathcal{F}$:

$$\int_A \xi(\omega)\, d\mathbb{P}(\omega) = \int_A \mathbb{E}(\xi|\mathcal{F})\, d\mathbb{P}(\omega)$$

$$\mathbb{E}(\xi; A) \qquad\qquad \mathbb{E}(\mathbb{E}(\xi|\mathcal{F}); A)$$



Examples. $\Omega=\{0,1\}^2$, $\mathcal{A}=\mathcal{P}(\Omega)$,

X_1 & X_2 are two indep coin toss $(X_1(\omega)=\omega_1)$, $Y=X_1+X_2$.

$\mathbb{E}(Y|X_1)(\omega)=:Z$, $\sigma(X_1)=\{\phi,\Omega,\{(0,1),(0,0)\},\{(1,0),(1,1)\}\}$

1. $Z\in\sigma(X_1) \Longleftrightarrow Z(0,1)=Z(0,0)=c_1,\ Z(1,0)=Z(1,1)=c_2$

2. $\int_A Z(\omega)d\mathbb{P}(\omega)=\int_A Y d\mathbb{P}(\omega)\quad \forall A\in\sigma(X_1)$

$A_0=\{(0,1),(0,0)\}$, $A_1=\{(1,0),(1,1)\}$

(i) $\int_{A_0} Y(\omega)d\mathbb{P}(\omega)=\mathbb{P}(\{(0,1)\})\cdot 1+\mathbb{P}(\{(0,0)\})\cdot 0=1/4 \qquad \int_{A_1} Y(\omega)d\mathbb{P}(\omega)=3/4$

(ii) $\int_{A_0} Z(\omega)d\mathbb{P}(\omega)=\mathbb{P}(\{(0,1)\})\cdot c_1+\mathbb{P}(\{(0,0)\})\cdot c_1=\frac{1}{2}\cdot c_1$

(iii) $\int_{A_1} Z(\omega)d\mathbb{P}(\omega)=\mathbb{P}(\{(1,0)\})\cdot c_2+\mathbb{P}(\{(1,1)\})\cdot c_2=\frac{1}{2}\cdot c_2$

$$Z(\omega)=\begin{cases}\frac{1}{2}, & X_1(\omega)=0\\ \frac{3}{2}, & X_1(\omega)=1\end{cases}$$

Existence & uniqueness
measure theory recap.

Definition 3.1

On measurable space $(\Omega, \mathcal{F})$:

1. μ is a **measure**, if
   - $\mu : \mathcal{F} \to [0, \infty], \mu(\emptyset) = 0$.
   - If $E = \bigcup_{i=1}^{\infty} E_i$ disjoint union, then $\mu(E) = \sum_{i=1}^{\infty} \mu(E_i)$.
2. ν is a σ-**finite measure** if there exist sets $A_n \in \mathcal{F}$, s.t.
   - $\Omega = \bigcup_{n=1}^{\infty} A_n$
   - $\nu(A_n) < \infty$.

Definition 3.2 (Signed measure)

Given a measurable space $(\Omega, \mathcal{F})$ (Ω is a set, on which $\mathcal{F}$ is a σ-algebra). α is a signed measure on $(\Omega, \mathcal{F})$, if

- $\alpha(E) \in (-\infty, \infty]$, $\forall E \in \mathcal{F}$.
- $\alpha(\emptyset) = 0$.
- If $E = \bigcup E_i$ is disjoint union, then $\alpha(E) = \sum_i \alpha(E_i)$, in such sense, that
  1. If $\alpha(E) < \infty$, then there is absolute convergence,
  2. If $\alpha(E) = \infty$, then $\sum_i \alpha(E_i)^- < \infty$ and $\sum_i \alpha(E_i)^+ = \infty$.

Jordan's Theorem: $\exists \alpha_1, \alpha_2$ are positive measures, that $\alpha_1 \perp \alpha_2$ and $\alpha = \alpha_1 - \alpha_2$.

Absolute continuity.

Let

- μ be a finite or σ-finite measure on $\mathcal{F}$
- ν be a finite, signed measure on $\mathcal{F}$.

We say that measure ν is absolute continuous for μ ($\nu \ll \mu$), if

$$\forall C \in \mathcal{F} : \mu(C) = 0 \Rightarrow \nu(C) = 0.$$

Remark. If ν has a density wrt. μ,

i.e. $\exists f \in \mathcal{L}_\infty(\mu) \cap \mathcal{L}_1(\mu)$ s.t. $\nu(C) = \int_C f(x)\, d\mu(x) \ \forall C \in \mathcal{F}$.

$$|\nu(C)| = \left|\int_C f(x)\, d\mu(x)\right| \leq \int_C |f(x)|\, d\mu(x) \leq \mu(C) \cdot \|f\|_\infty$$

(true for $f \in \mathcal{L}^1$ too!)

Theorem 3.3 (Radon-Nikodym)

1. $(\Omega, \mathcal{F})$ *probability space.*
2. μ σ-finite, (enough: μ is finite, for now)

2.5 ν a signed measure on $\mathcal{F}$ (we can think it is finite as well)

3. $\nu \ll \mu$ on $\mathcal{F}$.

Then $\exists f \in \mathcal{F}$, s.t.

(a) $\int |f(\omega)| d\mu(\omega) < \infty$,

(b) $\nu(C) = \int_C f(\omega) d\mu(\omega)$, $\forall C \in \mathcal{F}$.

(c) *If $f_1, f_2 \in \mathcal{F}$ satisfy (a) and (b), then* $f_1(\omega) = f_2(\omega)$ for μ a.e. $\omega \in \Omega$ ($\Rightarrow \nu$ a.e. $\omega \in \Omega$)

<u>Notation</u>: $f = \frac{\partial \nu}{\partial \mu}$, is the R-N derivative of ν wrt μ.

R-N THM.

$(\Omega, \mathcal{F})$ measurable space

- μ is a finite measure on it
- ν is a finite signed measure on it
- $\nu \ll \mu$

Then: $\exists f \in \mathcal{F}$

A1 $\int |f(\omega)| d\mu(\omega) < \infty$ $(f \in \mathcal{L}^1(\mu))$

B1 $\forall C \in \mathcal{A}$ $\nu(C) = \int_C f(\omega) d\mu(\omega)$

C1 f is μ & ν a.s. unique.

C.E. DEF.

$(\Omega, \mathcal{A}, \mathbb{P})$, $\mathcal{F}$ is a sub-σ-alg. of $\mathcal{A}$.
$\xi \in \mathcal{A}$, $\int |\xi| d\mathbb{P} < \infty$. $\mathcal{Z} = \mathbb{E}(\xi | \mathcal{F})$
if $\mathcal{Z} \in \mathcal{F}$

A2 / $\mathcal{Z} \in \mathcal{L}^1(\mathbb{P})$

B2 / $\forall C \in \mathcal{A}$ $\int_C \xi(\omega) d\mathbb{P}(\omega) = \int \mathcal{Z}(\omega) d\mathbb{P}(\omega)$;

Choose $\mu(A) = \mathbb{P}(A)$ $\forall A \in \mathcal{A}$

$\nu_\xi(A) = \int_A \xi(\omega) d\mathbb{P}(\omega) = \mathbb{E}(\xi; A)$

then ν_ξ is a signed measure on $\mathcal{A}$; $\nu_\xi \ll \mathbb{P}$ on $\mathcal{A}$ $\Rightarrow$ we can restrict both to $\mathcal{F} \subseteq \mathcal{A}$: $\nu_{\xi|\mathcal{F}}$ is a signed m. on $\mathcal{F}$, $\nu_{\xi|\mathcal{F}} \ll \mathbb{P}_{|\mathcal{F}}$ $\Rightarrow \exists f \in \mathcal{F}$ sat. **A1, B1, C1**

$\mathcal{Z} = f$ sat **A2** (from **A1**) & **B2** (from **B1**) & is unique up to $\mathbb{P}_{|\mathcal{F}}$. //

* then ν_ξ is a (finite) signed measure on $\mathcal{A}$, for $A \in \mathcal{A}$

- $$|\nu_\xi(A)| = \left|\int_A \xi(\omega)\,d\mathbb{P}(\omega)\right| \leq \int_A |\xi(\omega)|\,d\mathbb{P}(\omega) \leq \int_\Omega |\xi(\omega)|\,d\mathbb{P}(\omega) < \infty$$
- only takes values on $(-\infty,\infty) \subseteq (-\infty,\infty]$
- $$\nu_\xi\Big(\bigcup_{i=1}^\infty A_i\Big) = \int_{\bigcup_{i=1}^\infty A_i} \xi(\omega)\,d\mathbb{P}(\omega) = \sum_{i=1}^\infty \int_{A_i} \xi(\omega)\,d\mathbb{P}(\omega)$$

cons. of Leb. dom. conv. for the function $f_k(x) = \mathbb{1}\{\omega \in \bigcup_{i=1}^k A_i\} \cdot \xi(\omega)$

$$\lim_{k\to\infty} f_k(\omega) = \mathbb{1}\{\omega \in A\} \cdot \xi(\omega)$$

+ abs. conv. follows from dom conv of

$$\sum_{i=1}^\infty \int_{A_i} |\xi(\omega)|\,d\mathbb{P}(\omega) = \int_\Omega \sum_{i=1}^\infty \xi(\omega) \cdot \mathbb{1}\{\omega \in A_i\}\,d\mathbb{P}(\omega)$$

Uniqueness

We call the Radon-Nikodym derivative $\frac{d\nu_{\xi|\mathcal{F}}}{d\mathbb{P}_{|\mathcal{F}}}$ as a version of $\mathbb{E}(\xi|\mathcal{F})$.

Conditional Probability

Definition 4.1 (Conditional probability)

Let $\mathcal{F}$ be a sub-σ-algebra of $\mathcal{A}$. For every $A \in \mathcal{A}$ conditional probability of A with respect to (w.r.t.) the σ-algebra $\mathcal{F}$:

$$\mathbb{P}(A|\mathcal{F}) := \mathbb{E}[\mathbb{1}_A|\mathcal{F}]. \tag{11}$$

A way to think about it:

$(\Omega, \mathcal{A}, \mathbb{P})$ prob. space.

$\xi \in \mathcal{A}$,

$\mathcal{F}$ is a sub-σ-algebra of $\mathcal{A}$ $\mathbb{E}(\xi|\mathcal{F})$ is the information we have of ξ, when we are looking through a „camera with resolution $\mathcal{F}$".

E.g.  & 

$\mathbb{E}(\xi|\mathcal{A}) = \xi$

$\mathbb{E}(\xi|\mathcal{F})$

Properties:

a) Linearity:

$$\underbrace{\mathbb{E}(a\cdot X + bY \mid \mathcal{F})}_{LHS} = \underbrace{a\cdot \mathbb{E}(X\mid\mathcal{F}) + b\,\mathbb{E}(Y\mid\mathcal{F})}_{RHS}$$

Proof. Idea: Check whether RHS satisfies the def of conditional exp. for the random variable $a\cdot X + bY$.

① $a\cdot\mathbb{E}(X\mid\mathcal{F}) + b\cdot\mathbb{E}(Y\mid\mathcal{F}) \in \mathcal{F}$

② $a\cdot\mathbb{E}(X\mid\mathcal{F}) + b\cdot\mathbb{E}(Y\mid\mathcal{F}) \in \mathcal{L}^1(\mathbb{P})$

③ $$\int_A a\cdot\mathbb{E}(X\mid\mathcal{F}) + b\cdot\mathbb{E}(Y\mid\mathcal{F})\,d\mathbb{P} = a\underbrace{\int_A \mathbb{E}(X\mid\mathcal{F})\,d\mathbb{P}}_{\int_A X\,d\mathbb{P}} + b\cdot\underbrace{\int_A \mathbb{E}(Y\mid\mathcal{F})\,d\mathbb{P}}_{\int_A Y\,d\mathbb{P}} = \int_A aX + bY\,d\mathbb{P}$$

b) Monotonicity:

$$X \le Y \Rightarrow \mathbb{E}(X|\mathcal{F}) \le \mathbb{E}(Y|\mathcal{F})$$

<u>Pr.</u> Assume that $\exists G \in \mathcal{F}$ s.t. $\mathbb{P}(G) > 0$ & on G we have $\mathbb{E}(X|\mathcal{F})(g) > \mathbb{E}(Y|\mathcal{F})(g)$ $\forall g \in G$

$$0 \underset{\text{linearity}}{<} \int_G \mathbb{E}(X-Y|\mathcal{F})(g)\, d\mathbb{P}(g) \underset{\text{3rd prop. of cond. exp.}}{=} \int_G (X-Y)(g)\, d\mathbb{P}(g) \le 0 \;\lightning$$

Properties continued

spec.: tower property

(c) **Csebisev inequality**:

$$\mathbb{P}\left(|X| \geq a|\mathcal{F}\right) \leq a^{-2}\mathbb{E}\left[X^2|\mathcal{F}\right]. \tag{12}$$

(d) **Monoton convergence theorem**: Let us assume, that $X_n \geq 0,\ X_n \uparrow X,\ \mathbb{E}[X] < \infty$ then

$$\mathbb{E}[X_n|\mathcal{F}] \uparrow \mathbb{E}[X|\mathcal{F}].$$

(e) Applying the above for $Y_1 - Y_n$: If $Y_n \downarrow Y$, $\mathbb{E}[|Y_1|], \mathbb{E}[|Y|] < \infty$, then $\mathbb{E}[X_n|\mathcal{F}] \downarrow \mathbb{E}[X|\mathcal{F}]$.

(f) **Jensen inequality**: If φ is convex, $\mathbb{E}[|X|], \mathbb{E}[|\varphi(X)|] < \infty$, then

$$\varphi(\mathbb{E}[X|\mathcal{F}]) \leq \mathbb{E}[\varphi(X)|\mathcal{F}]. \tag{13}$$

(g) **Conditional Cauchy Schwarz:**

$$\mathbb{E}[XY|\mathcal{F}]^2 \leq \mathbb{E}\left[X^2|\mathcal{F}\right]\mathbb{E}\left[Y^2|\mathcal{F}\right]. \tag{14}$$

(h) $X \to \mathbb{E}[X|\mathcal{F}]$ is a contraction on L^p, if $p \geq 1$:

$$\mathbb{E}[|\mathbb{E}[X|\mathcal{F}]|^p] \leq \mathbb{E}[|X|^p]$$

(i) If $\mathcal{F}_1 \subset \mathcal{F}_2$, then

1. $\mathbb{E}[\mathbb{E}[X|\mathcal{F}_1]|\mathcal{F}_2] = \mathbb{E}[X|\mathcal{F}_1]$
2. $\mathbb{E}[\mathbb{E}[X|\mathcal{F}_2]|\mathcal{F}_1] = \mathbb{E}[X|\mathcal{F}_1]$

So always the more primitive σ-algebra wins.

(j) If $X \in \mathcal{F}$, $\mathbb{E}[|Y|], \mathbb{E}[|XY|] < \infty$, then

$$\mathbb{E}\left[X \cdot Y|\mathcal{F}\right] = X \cdot \mathbb{E}[Y|\mathcal{F}]. \tag{15}$$

(k) **$\mathbb{E}[X|\mathcal{F}]$ as projection**: Let us assume, that $\mathbb{E}\left[X^2\right] < \infty$. Then $\mathbb{E}[X|\mathcal{F}]$ is the orthogonal projection of X to $L^2(\Omega, \mathcal{F}, \mathbb{P})$. In other words:

$$\mathbb{E}\left[(X - \mathbb{E}[X|\mathcal{F}])^2\right] = \min_{Y \in \mathcal{F}} \mathbb{E}\left[(X - Y)^2\right].$$

(l) **$X \to \mathbb{E}[X|\mathcal{F}]$ is self-adjoint on $L^2(\Omega, \mathcal{A}, \mathbb{P})$:**

$$\begin{aligned} \mathbb{E}[X \cdot \mathbb{E}[Y|\mathcal{F}]] &= \mathbb{E}[\mathbb{E}[X|\mathcal{F}] \cdot \mathbb{E}[Y|\mathcal{F}]] \\ &= \mathbb{E}[\mathbb{E}[X|\mathcal{F}] \cdot Y]. \end{aligned} \tag{16}$$

spec. $\mathbb{E}(X|\mathcal{F}) \in \mathcal{L}^1(\mathbb{P})$

Properties of conditional expectation V

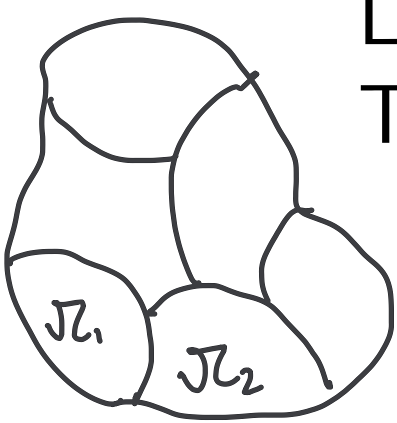
Let us define conditional variation w.r.t. σ-algebra (see [1, Def. 7.35] and [1, Statement 7.36]):

$$\mathrm{Var}(X|\mathcal{F}) := \mathbb{E}\left[X^2|\mathcal{F}\right] - \mathbb{E}\left[X|\mathcal{F}\right]^2.$$

Then

(m) $\boxed{\mathrm{Var}\,(X) = \mathbb{E}\left[\mathrm{Var}(X|\mathcal{F})\right] + \mathrm{Var}\left(\mathbb{E}\left[X|\mathcal{F}\right]\right).}$

(n) $\Omega = \bigcup\limits_{i=1}^{\infty} \Omega_i$ is disjoint union and $\mathbb{P}(\Omega_i) > 0$.
Let $\mathcal{F}$ be the σ-algebra generated by $\{\Omega_i\}_{i=1}^{\infty}$.
Then for a r.v. X:



$$\mathbb{E}\left[X|\mathcal{F}\right] = \sum_i \frac{\mathbb{E}\left[X;\Omega_i\right]}{\mathbb{P}(\Omega_i)} \cdot \mathbb{1}_{\Omega_i}.$$

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Properties of conditional expectation VI

(p) **Bayes's formula:** Let $F \in \mathcal{F}$ and $A \in \mathcal{A}$. Then

$$\mathbb{P}\left(F|A\right) = \frac{\int\limits_F \mathbb{P}(A|\mathcal{F})}{\int\limits_\Omega \mathbb{P}(A|\mathcal{F})}. \tag{17}$$

Is is easy to see, that this statement gives Bayes-theorem, in the case, when $\mathcal{F}$ is generated by a partition.

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Heu. intro for martingales

Definition 5.1

- An increasing sequence of σ-algebras $\mathcal{F}_n$ is called **filtration**.
- X_n is **adapted** to $\mathcal{F}_n$, if $X_n \in \mathcal{F}_n$, $\forall n$.
- (X_n) is a **martingale** for filtration $\mathcal{F}_n$, if

  (a) $\mathbb{E}\left[|X_n|\right] < \infty$

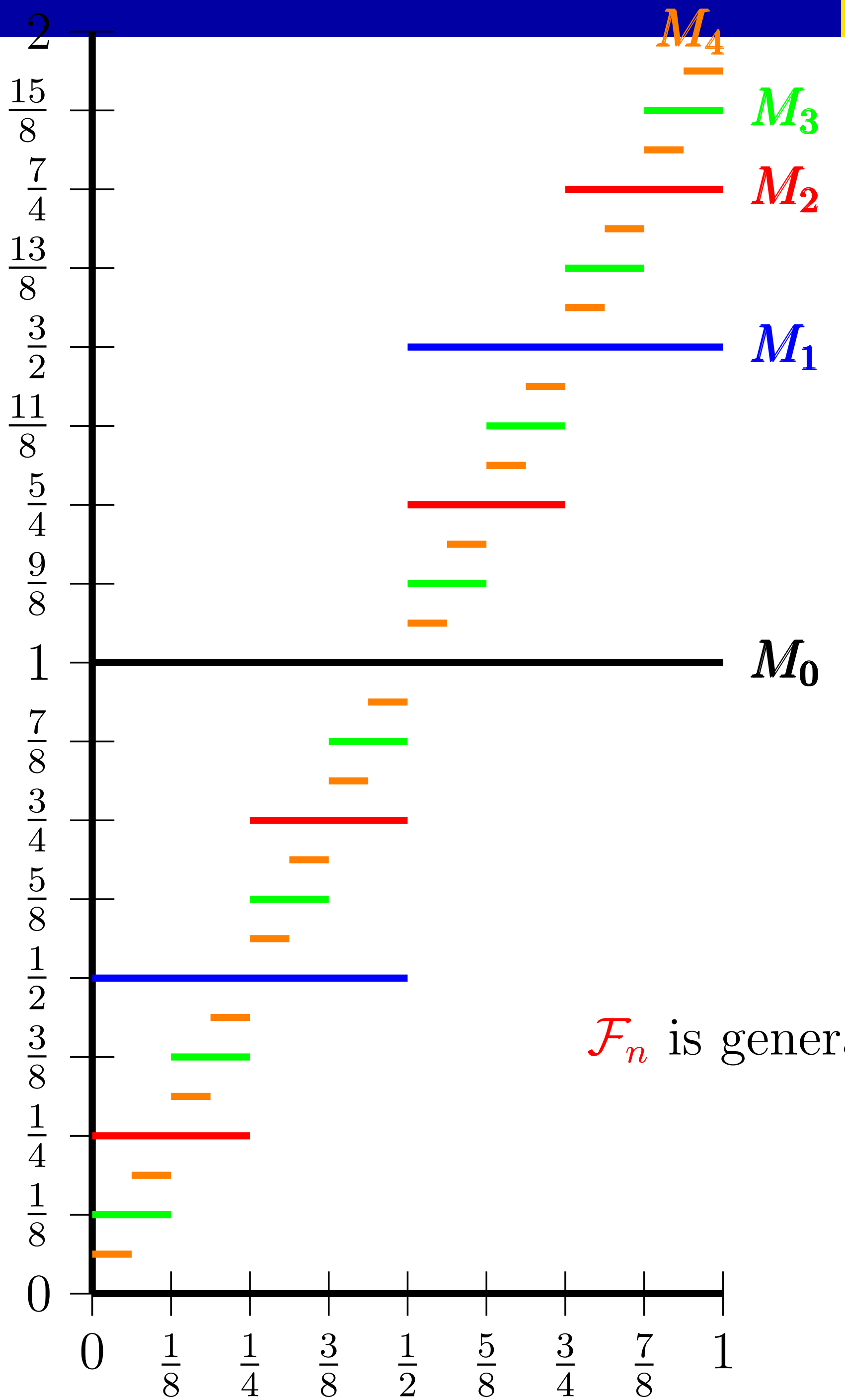
  (b) X_n is adapted to $\mathcal{F}_n$,

  (c) $\mathbb{E}(X_{n+1}|\mathcal{F}_n) = X_n$, $\forall n \geq 1$.

If (a) and (b) are satisfied, but $=$ of (c) is replaced by

(c') $\leq$, then (X_n) is a **supermartingale**,

(c'') $\geq$, then (X_n) is a **submartingale**.



$\Omega = [0,1]$ and $\mathbb{P} = \mathcal{L}\mathrm{eb}|_{[0,1]}$

if $x = \sum_{n=1}^{\infty} x_n 2^{-n}$, $x_n \in \{0,1\}$

let $\theta_k = \begin{cases} 1, & \text{if } x_k = 1; \\ -1, & \text{if } x_k = 0. \end{cases}$

From this, $M_n(x) = 1 + \sum_{k=1}^{n} \theta_k 2^{-k}$

$\mathcal{F}_n$ is generated by $\left\{ \left[\frac{i}{2^n}, \frac{i+1}{2^n}\right) : i = 0, \ldots 2^n - 1 \right\}$

$M_n \in \mathcal{F}_n$

$\mathbb{E}\left[M_{n+1} | \mathcal{F}_n\right] = M_n$

Example 5.3

We throw a regular coin many times. Let the outcome of the n^{th} throw be $\xi_n = 1$ if it's head and $\xi_n = -1$ if it's tail. Let $X_n := \xi_1 + \cdots + \xi_n$ and $\mathcal{F}_n := \sigma\{\xi_1, \ldots, \xi_n\}$ if $n \geq 1$ and $X_0 = 0$ and $\mathcal{F}_0 = \{\emptyset, \Omega\}$.

- $\mathbb{E}(|X_n|) \leq n$
- $X_n \in \mathcal{F}_n$, clearly &
- $\mathbb{E}[X_{n+1}|\mathcal{F}_n] = \underbrace{\mathbb{E}[X_n|\mathcal{F}_n]}_{X_n} + \underbrace{\mathbb{E}[\xi_{n+1}|\mathcal{F}_n]}_{0} = X_n$.

So X_n is a martingale for $\mathcal{F}_n$.

Generally

Example 5.2

Let us imagine a player, who plays a fair game (with expected value 0) very many times. Let M_n be his/her winning after the n^{th} game (or losing if M_n is negative) and let Y_n be the outcome of the n^{th} game and let $\mathcal{F}_n = \sigma(Y_1, \ldots, Y_n)$. Then (M_n) is a martingale for $\mathcal{F}_n$.

Example 5.4

Let $X_1, \ldots, X_n$ i.i.d. $\mathbb{E}\left[X_i\right] = \mu$ and $S_n := S_0 + X_1 + \cdots + X_n$ be a random walk. Then $M_n := S_n - n\mu$ is a martingale for $\mathcal{F}_n := \sigma(X_1, \ldots, X_n)$.
Namely: $M_{n+1} - M_n = X_{n+1} - \mu$ is independent of $X_n, \ldots, X_1, S_0$, so

$$\mathbb{E}\left[M_{n+1} - M_n | \mathcal{F}_n\right] = \mathbb{E}\left[X_{n+1}\right] - \mu = 0.$$

So

$$\mathbb{E}\left[M_{n+1} | \mathcal{F}_n\right] = M_n.$$

If $\mu \leq 0$, then S_n supermartingale and if $\mu \geq 0$, then S_n submartingale.

Theorem 5.5

Let X_n be a MC, whose transition matrix is $\mathbf{P} = (p(x,y))_{x,y}$. Let us assume, that for a function $f : S \times \mathbb{N} \to \mathbb{R}$:

$$f(x,n) = \sum_y p(x,y) f(y,n+1). \tag{18}$$

Then $M_n = f(X_n, n)$ is a martingale for $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$. In the special, when

$$\boxed{h(x) = \sum_y p(x,y) h(y)}, \tag{19}$$

then $h(X_n)$ is martingale for $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$.

Proof.

$$\begin{aligned}\mathbb{E}\left[f(X_{n+1}, n+1) | \mathcal{F}_n\right] &= \sum_y p(X_n, y) f(y, n+1) \\ &= f(X_n, n).\end{aligned}$$

□

Example 5.6 (Gambler's Ruin)

Let $X_1, X_2, \ldots$ i.i.d. s.t. for some $p \in (0,1)$, $p \neq 1/2$:

$$\mathbb{P}(X_i = 1) = p \text{ and } \mathbb{P}(X_i = -1) = q = 1 - p.$$

Let $S_n = S_0 + X_1 + \cdots X_n$. Then

$$M_n := \left(\frac{q}{p}\right)^{S_n}$$

is a martingale.

This comes from that $h(x) = \left(\frac{q}{p}\right)^x$ satisfies condition $h(x) = \sum_y p(x,y)h(y)$) apply Theorem 5.5.

Example 5.7 (Simple symmetric random walk)

$Y_1, Y_2, \ldots$ i.i.d. $\mathbb{P}(Y_i = 1) = \mathbb{P}(Y_i = -1) = 1/2$. $S_n = S_0 + Y_1 + \cdots + Y_n$. Then $M_n := S_n^2 - n$ is a martingale for $\sigma(Y_1, \ldots, Y_n)$.

Namely: we must show, that for $f(x, n) = x^2 - n$ the equality in (18) is satisfied. In other words, that

$$x^2 - n = \frac{1}{2}((x-1)^2 - (n+1)) + \frac{1}{2}\left((x+1)^2 - (n+1)\right).$$

And this is given by a trivial computation.

(18) $f(x, n) = \sum_y p(x, y) f(y, n+1)$.

Example 5.8 (product of independent r.v.s)

Given are $X_1, X_2, \dots \geq 0$ i.i.d. and $\mathbb{E}[X_i] = 1$. Then $M_n = M_0 \cdot X_1 \cdots X_n$ is a martingale for $\mathcal{F}_n := \sigma(X_1, \dots, X_n)$.

Namely:

$$\mathbb{E}[M_{n+1} - M_n | \mathcal{F}_n] = M_n \cdot \mathbb{E}[X_{n+1} - 1 | \mathcal{F}_n] = 0.$$

This latter is because X_{n+1} is independent of $X_1, \dots, X_n$, hence X_{n+1} is also independent of the σ-algebra $\mathcal{F}_n$ generated by them. So, $\mathbb{E}[X_{n+1} - 1 | \mathcal{F}_n] = \mathbb{E}[X_{n+1} - 1] = 0$.

Theorem 5.9

Let $\varphi, \psi : \mathbb{R} \to \mathbb{R}$ a convex function and ψ be increasing.

- *(a) If M_n is a martingale, then $\varphi(M_n)$ is a submartingale.*
- *(b) If M_n submartingale, then $\psi(M_n)$ is a submartingale also.*

This is an immediate corollary of Jensen's inequality (formula (13)) and the definition.
So, if M_n is a martingale, then e.g. $|M_n|$ and M_n^2 are submartingale.

$$\mathbb{E}(\varphi(M_{n+1}) | \mathcal{F}_n) \geq \varphi(\mathbb{E}(M_{n+1} | \mathcal{F}_n)) = \varphi(M_n)$$
$$\mathbb{E}(\psi(M_{n+1}) | \mathcal{F}_n) \geq \psi(\mathbb{E}(M_{n+1} | \mathcal{F}_n)) \geq \psi(M_n)$$

Theorem 5.10

Let M_n be a martingale. Then

$$\mathbb{E}\left[M_{n+1}^2|\mathcal{F}_n\right] - M_n^2 = \mathbb{E}\left[(M_{n+1} - M_n)^2|\mathcal{F}_n\right]. \tag{20}$$

Proof.

$$\mathbb{E}\left[(M_{n+1} - M_n)^2\,|\mathcal{F}_n\right] = \tag{21}$$

$$\mathbb{E}\left[M_{n+1}^2|\mathcal{F}_n\right] - 2M_n \underbrace{\mathbb{E}\left[M_{n+1}|\mathcal{F}_n\right]}_{M_n} + M_n^2$$

$$= \mathbb{E}\left[M_{n+1}^2|\mathcal{F}_n\right] - M_n^2.$$

□

Now we prove the orthogonality of the increments of the martingale.

Theorem 5.11

Let M_n be a martingale and let $0 \le i \le j \le k < n$. Then

$$\mathbb{E}\left[(M_n - M_k) \cdot M_j\right] = 0. \tag{22}$$

and its obvious corollary:

$$\mathbb{E}\left[(M_n - M_k) \cdot (M_j - M_i)\right] = 0. \tag{23}$$

Proof.

Proof of (22):

$$\begin{aligned}
\mathbb{E}\left[(M_n - M_k)M_j\right] &\overset{(i)}{=} \mathbb{E}\left[\mathbb{E}\left[(M_n - M_k)M_j | \mathcal{F}_k\right]\right] \\
&= \mathbb{E}\Big[M_j \cdot \underbrace{\mathbb{E}\left[(M_n - M_k) | \mathcal{F}_k\right]}_{0}\Big] = 0
\end{aligned}$$

$M_j \in \mathcal{F}_k$ & (j)

Corollary 5.12

Using notation of Theorem 5.11:

$$\mathbb{E}\left[(M_n - M_0)^2\right] = \sum_{k=1}^{n} \mathbb{E}\left((M_k - M_{k-1})^2\right)$$

Proof.

By using formula (23):

$$\mathbb{E}\left[(M_n - M_0)^2\right] = \mathbb{E}\left[\left(\sum_{k=1}^{n} M_k - M_{k-1}\right)^2\right]$$

$$= \sum_{k=1}^{n} \mathbb{E}\left[(M_k - M_{k-1})^2\right]$$

$$+ 2 \sum_{1 \leq j < k \leq n} \underbrace{\mathbb{E}\left[(M_k - M_{k-1})(M_j - M_{j-1})\right]}_{0}.$$

Let $m \leq n$, then from the definition:

Lemma 5.13

- *If M_n is martingale, then $\mathbb{E}[M_m] = \mathbb{E}[M_n]$,*
- *If M_n is submartingale, then $\mathbb{E}[M_m] \leq \mathbb{E}[M_n]$,*
- *If M_n is supermartingale, then $\mathbb{E}[M_m] \geq \mathbb{E}[M_n]$.*

The next example is about a famous betting strategy. Then we will see that

$$\text{"you can't beat an unfavorable game."} \tag{24}$$