Dimension of overlapping random Cantor sets

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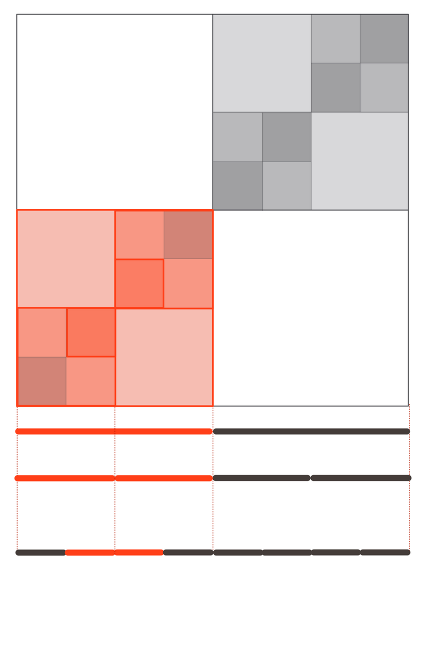
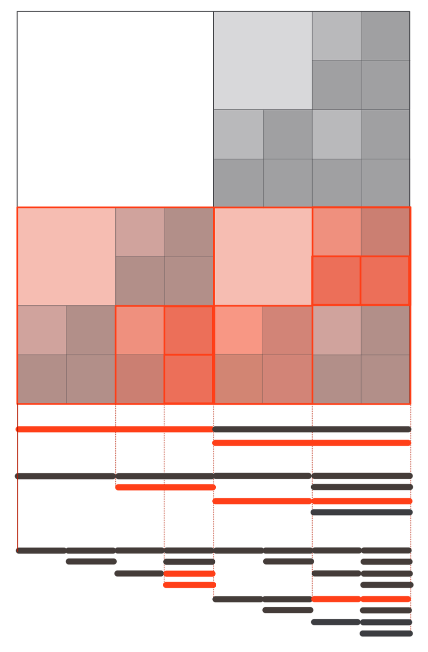
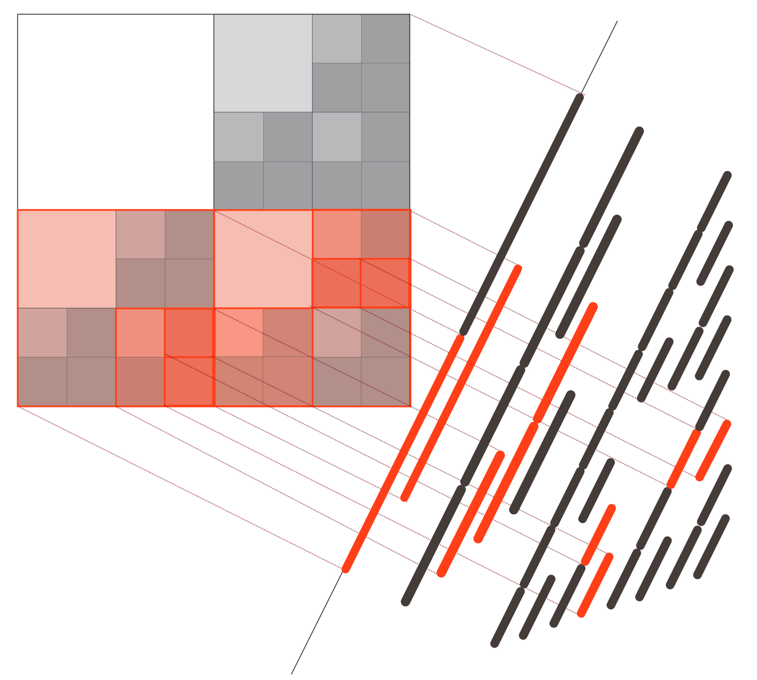
Abstract.

Using a similar random process to the one which yields the Mandelbrot percolation sets, we can define overlapping Mandelbrot percolation sets. We investigate the Hausdorff dimension of these sets.

Random sets on IFSs.

Fix a self-similar IFS $\mathcal{S} = \{S_i\}_{i=1}^{M}$ on the line and a compact interval I with $S_i(I) \subseteq I$ for all i. For a parameter $p \in (0, 1]$, we run a percolation process on the cylinders: at the first level, each $S_i(I)$ is retained independently with probability p; inside every retained cylinder, we repeat the process on its M subcylinders, and so on. The random attractor Λ_p is the limit set of the surviving cylinders.

Known results: Hausdorff dimension of some percolation self-similar sets.

$S_i(x) = (x + t_i)/L$, $2 \le L \in \mathbb{N}$, $t_i \in \{0, \ldots, L-1\}$.		$S_i(x) = (x + t_i)/L$, $t_i \in \mathbb{Q}$.
		
Negligible overlaps: $t_i \neq t_j$ for $i \neq j$. $\dim_H \Lambda_p = \frac{\log(Mp)}{\log(L)}$ see e.g. [1].	**Negligible or trivial overlaps:** no restriction on t_i. $\inf\limits_{t\in[0,1]} \frac{t\log(p)+\log\left(\sum_{i=0}^{L-1} \#\{i:t_i=k\}^t\right)}{\log(L)}$ see [2,3].	**Grid aligned (rational) overlaps**: $\dim_{\mathsf{H}} \Lambda_p = \dim_{\mathsf{B}} \Lambda_p = \dim_{\mathsf{P}} \Lambda_p$, by [4]. For the value see Theorem ↗.

Preliminaries.

For a deterministic IFS of the form:

$$\mathcal{S} = \left\{ S_i(x) = \frac{x}{L} + t_i \right\}_{i=1}^{M}, \quad L \in \mathbb{N},\ t_i \in \mathbb{Q}$$

we can associate a set of non-negative matrices $\mathcal{B} := \{\mathbf{B}_0, \ldots, \mathbf{B}_{L-1}\}$.
The *subadditive pressure* corresponding to the non-negative matrices $\mathcal{B}$ is defined as

$$P(t) := \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{\underline{\theta}\in\{0,\ldots,L-1\}^n} \|\mathbf{B}_{\underline{\theta}}\|^t \right),$$

the limit exists for all $t \in \mathbb{R}$. By [5, 6] $P(t)$ is continuously differentiable & convex for $t > 0$.
The *Lyapunov exponent* λ corresponding to the non-negative matrices $\mathcal{B}$ is the constant satisfying

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \log(\|\mathbf{B}_{\underline{\theta}|_n}\|),$$

for almost every $\underline{\theta} \in \{0, \ldots, L-1\}^{\mathbb{N}}$ with respect to the uniform measure.

Theorem [O.–Simon].

Consider the randomized IFS $\mathcal{S}$, with attractor Λ_p and the associated set of matrices $\mathcal{B} = \{\mathbf{B}_0, \ldots, \mathbf{B}_{L-1}\}$. We assume, that

I. each matrix has a positive element in every row and column,
II. there exists a product $\mathbf{B}_{i_1} \cdots \mathbf{B}_{i_k}$ which is a strictly positive.

Then

$$\dim_{\mathsf{H}} \Lambda_p = \dim_{\mathsf{B}} \Lambda_p = \dim_{\mathsf{P}} \Lambda_p = \inf_{t\in[0,1]} \frac{P(t) + t\log(p)}{\log(L)} = \begin{cases} 1, & \lambda \ge 0 \\ \log(Mp)/\log(L), & P'(1) \le 0 \\ \inf_{t\in[0,1]} \frac{P(t)+t\log(p)}{\log(L)}, & \text{otherwise,} \end{cases}$$

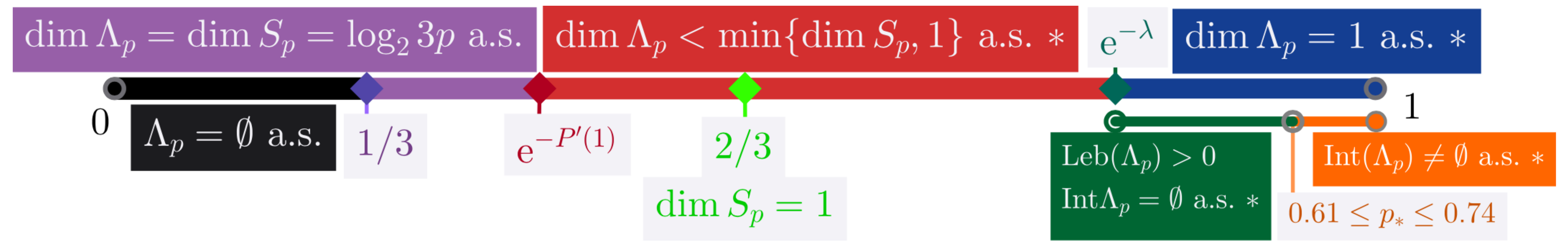
almost surely conditioned on non-extinction.

*Remarks.

The results hold for a more general random model based on labelled Galton-Watson trees.
The results hold in dimension d: Run the percolation on the cylinders of IFSs of the following form: $\mathcal{S} := \left\{S_i(x) := \frac{1}{L}\mathbf{x} + \mathbf{t}_i\right\}_{i=1}^{K}$, $S_i : \mathbb{R}^d \to \mathbb{R}^d$, where: $2 \le L \in \mathbb{N}$, $\mathbf{t}_i \in \mathbb{N}^d$, $\exists h \in \mathbb{N}$, $L-1|h$, $\forall j \in [d]\ 0 = \min_i \mathbf{t}_i(j)$, $h = \max_i \mathbf{t}_i(j)$.

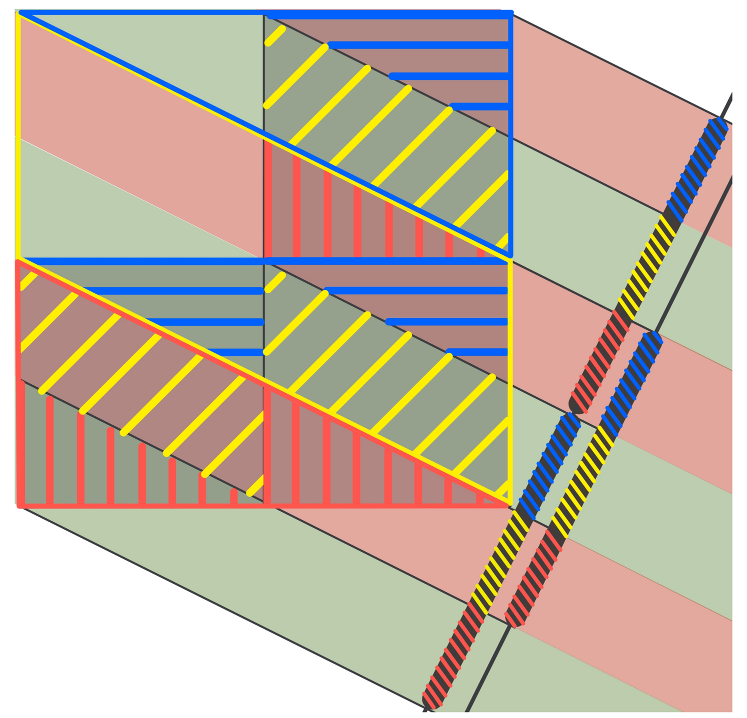
Randomized 0-1-3, with contraction ratio $1/2$.

Λ_p is the percolation on the cylinders of the IFS $\{x/2, x/2+1, x/2+3\}$, the projection of the random right-angled Sierpiński carpet (S_p). See Matrices ↓.



The parameter intervals and phase transitions as we vary p.

Matrices.



$$\mathbf{B}_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad \mathbf{B}_1 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

References.

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